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LOCALIZATION PROPERTIES OF A FRICTIONAL MATERIAL MODEL BASED ON REGULARIZED STRONG DISCONTINUITY

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SUMMARY

A frictional material model is investigated with respect to the existence and development of a regularized strong discontinuity within the constitutive framework of plasticity theory. It appears that the condition for the existence of such a discontinuity is identical to the classical bifurcation criterion for band shaped bifurcation in the rate of deformation field. The model behaviour is also discussed beyond onset of localization, for a band with fixed orientation, where the state variables are allowed to change.

KEY WORDS: localization; regularization; discontinuity; plasticity; mohr-Coulomb; ductile fracture

INTRODUCTION

It is a well documented fact that the conventional rate-independent continuum theory leads to pathological FE-mesh sensitivity in the presence of non-classical properties (such as strain-softening and/or non-associated flow rule). Such sensitivity of the mesh design relates to the size of elements as well as to the orientation of element sides. The problem is attributed to ill-posedness of the incremental problem at the state where ellipticity is lost, which is obtained if the FE approximation of the displacement field is assumed to be continuous throughout the body. In order to remedy this pathologic behavior, various 'continuum regularizations' have been proposed in the literature, such as micro-polar (or Cosserat) theory,[1] and non-local continuum theory.[2]

Upon introducing a narrow localization band, across which a strong discontinuity is regularized, it was shown by Larsson *et al.*,[3] and more recently by Simo *et al.*,[4] Larsson and Runesson,[5] that the spurious mesh sensitivity for the classical continuum can be avoided. As a result, the constitutive properties of the localization band are determined, in an automatic fashion, by the choice of (local) material law. Like in the classical bifurcation criterion,[6,7] the existence of a regularized strong discontinuity depends on the condition that characterizes the singularity of the acoustic tensor.

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In this paper, we evaluate the behaviour of a frictional material model, within the plasticity framework, along the lines set out in Reference 5. The model is based on the Mohr–Coulomb criterion together with a non-associated flow rule, and it is assessed with regard to the development of band shaped localization in the context of a regularized strong discontinuity. The situation at onset of localization is thus determined by the condition that the acoustic tensor becomes singular for the first time. Moreover, the model behavior in the post-localized range is assessed with respect to the conditions that plastic loading is obtained inside the band, whereas elastic unloading takes place outside the band. To this end, it is assumed that the band orientation is fixed, whereas the state variables are allowed to change within the localization band. We consider, in particular, the sensitivity of the model for continued localization, when the principal stress axes rotates relative to the (fixed) band orientation.

CONSTITUTIVE EQUATIONS

The constitutive relation in small strain plasticity may be written

$$\dot{\boldsymbol{\varepsilon}} = \mathbf{C}^{e} : \dot{\boldsymbol{\sigma}} + \dot{\boldsymbol{\varepsilon}}^{p} \tag{1}$$

$$F(\boldsymbol{\sigma}, \varkappa) \leqslant 0, \qquad \dot{\lambda} \geqslant 0, \qquad F(\boldsymbol{\sigma}, \varkappa)\dot{\lambda} = 0 \tag{2}$$

where $\dot{\boldsymbol{\varepsilon}}$ and $\dot{\boldsymbol{\sigma}}$ are strain and stress rates, $\varkappa$ is the hardening/softening variable, λ is a plastic multiplier and $F = 0$ is the (convex) yield surface. The tensor $\mathbf{C}^e$ is the tensor of constant elastic flexibility, which is the inverse of the elastic stiffness modulus tensor $\mathbf{D}^e$. In order to define the evolution of plastic flow, we assume the non-associated flow rule

$$\dot{\boldsymbol{\varepsilon}}^{p} = \dot{\lambda}\mathbf{g}, \qquad \mathbf{g} = \frac{\partial G}{\partial \boldsymbol{\sigma}} \tag{3}$$

where the flow direction $\mathbf{g}$ is the gradient of the plastic potential $G(\boldsymbol{\sigma})$. It is also assumed that the hardening of the yield surface is represented only with a scalar $\varkappa$, which is defined by the rate law

$$\varkappa = k(\dot{\boldsymbol{\varepsilon}}^{p}) = \dot{\lambda}k(\mathbf{g}) \tag{4}$$

where k is a homogeneous (but in general non-linear) function of $\dot{\varepsilon}^p$.

By combining equations (1)–(4), the linearized response of the continuously deforming material can, formally, be described with the rate equation

$$\dot{\boldsymbol{\sigma}} = \begin{cases} \mathbf{D}^{ep} : \dot{\boldsymbol{\varepsilon}} & \text{if } \mathbf{f} : \mathbf{D}^{e} : \dot{\boldsymbol{\varepsilon}} > 0 \quad \text{(P)} \\ \mathbf{D}^{e} : \dot{\boldsymbol{\varepsilon}} & \text{if } \mathbf{f} : \mathbf{D}^{e} : \dot{\boldsymbol{\varepsilon}} \leqslant 0 \quad \text{(E)} \end{cases} \tag{5}$$

where (P) and (E) denote 'plastic' and 'elastic' loading, respectively. In equation (5), $\mathbf{f} = \partial \mathbf{F}/\partial \boldsymbol{\sigma}$, whereas $\mathbf{D}^{ep}$ is the continuum tangent stiffness modulus tensor pertinent to elastic–plastic response

$$\mathbf{D}^{ep} = \mathbf{D}^{e} - \frac{1}{h}\mathbf{D}^{e} : \mathbf{g}\mathbf{f} : \mathbf{D}^{e}, \qquad h = \mathbf{f} : \mathbf{D}^{e} : \mathbf{g} + H > 0 \tag{6}$$

where H is the hardening/softening modulus. As to H, we distinguish hardening ($H > 0$), perfect plasticity ($H = 0$) and softening ($H < 0$). In the following, we use the notation $\dot{\boldsymbol{\sigma}} = \mathbf{D} : \dot{\boldsymbol{\varepsilon}}$, where $\mathbf{D} = \mathbf{D}^{ep}$, in plastic loading, or $\mathbf{D} = \mathbf{D}^{e}$ corresponding to elastic (un)loading.

REGULARIZED DISCONTINUOUS FIELDS AND CONTINUITY EQUATIONS

We consider the scenario where all state variables, including the displacement $\mathbf{u}(\mathbf{x})$, are continuous fields until the state where localization is possible has been achieved. At this state, we consider changes of the state variables such that they are spatially smooth, except across the internal surface Γ_s (with unit normal $\mathbf{n}$), as shown in Figure 1. The necessary conditions for the existence of such a discontinuity are, therefore, established based on equilibrium and continuity considerations of the stress and strain fields, respectively. To this end, it is assumed that the solid (= fully drained soil) occupies the domain Ω with external boundary Γ, as shown in Figure 1. The discontinuity surface divides Ω into the subdomains Ω_- and Ω_+ in such a way that $\mathbf{n}$ is pointing from Ω_- to Ω_+.

We thus propose the decomposition of the velocity field $\dot{\mathbf{u}}(\mathbf{x})$ as

$$\dot{\mathbf{u}}(\mathbf{x}) = \dot{\mathbf{u}}_c(\mathbf{x}) + [\dot{\mathbf{u}}]H_s(\mathbf{x}) \tag{7}$$

where $H_s(\mathbf{x})$ is the Heaviside function, centered on Γ_s, and $[\dot{\mathbf{u}}]$ is the *constant* jump of $\dot{\mathbf{u}}(\mathbf{x})$ across Γ_s. This is defined as

$$[\dot{\mathbf{u}}] = \lim_{\varepsilon \to 0} [\dot{\mathbf{u}}(\mathbf{x}_0 + \varepsilon\mathbf{n}) - \dot{\mathbf{u}}(\mathbf{x}_0 - \varepsilon\mathbf{n})]; \quad \mathbf{x}_0 \in \Gamma_s \tag{8}$$

The Heaviside function, introduced in (7), is defined as

$$H_s(\mathbf{x}) = \begin{cases} 0 & \text{if } \mathbf{x} \in \Omega_- \\ 1 & \text{if } \mathbf{x} \in \Omega_+ \end{cases} \tag{9}$$

As to the gradient of $H_s(\mathbf{x})$, we introduce the Dirac delta distribution $\boldsymbol{\delta}_s(\mathbf{x})$ defined as

$$\int_\Omega \boldsymbol{\psi} \cdot \boldsymbol{\delta}_s \, d\Omega = \int_{\Gamma_s} \boldsymbol{\psi} \cdot \mathbf{n} \, d\Gamma = \int_\Omega \boldsymbol{\psi} \cdot \nabla H_s \, d\Omega, \quad \forall \boldsymbol{\psi} \in C_0^\infty(\Omega) \tag{10}$$

where the last equality was obtained by application of the divergence theorem. From (10), we now obtain, formally, that

$$\nabla H_s(\mathbf{x}) = \boldsymbol{\delta}_s(\mathbf{x}) \tag{11}$$

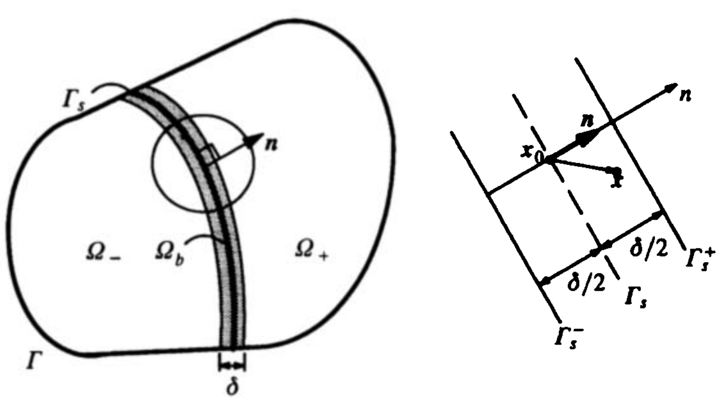

Figure 1. Solid with regularized discontinuity across Γ_s.

which has a meaning only is a distributional sense. A regularized version of $\boldsymbol{\delta}_s(\mathbf{x})$ is obtained by introducing a narrow band zone Ω_b (b = band) along Γ_s with the width δ, as shown in Figure 1, where the regularization function $\boldsymbol{\delta}_{s(\delta)}(\mathbf{x})$ is defined as

$$\boldsymbol{\delta}_{s(\delta)}(\mathbf{x}) \equiv \boldsymbol{\delta}_{s(\delta)}(\mathbf{x}_0, n) = \begin{cases} \dfrac{1}{\delta}\mathbf{n}(\mathbf{x}_0) & \text{iff } \mathbf{x} \in \Omega_b \\ 0 & \text{otherwise} \end{cases} \tag{12}$$

where $\mathbf{x} = \Omega_b$ is defined, for each $\mathbf{x}_0 \in \Gamma_s$ and $\mathbf{n} \geqslant 0$, such that $\mathbf{x} = \mathbf{x}_0 \pm n\mathbf{n}(\mathbf{x}_0)$. Note that the regularized Dirac delta distribution has the desired property that $\boldsymbol{\delta}_{s(\delta)}(\mathbf{x}) \to \boldsymbol{\delta}_s(\mathbf{x})$ in the sense that

$$\int_\Omega \boldsymbol{\psi}\cdot\boldsymbol{\delta}_{s(\delta)}\,\mathrm{d}\Omega = \frac{1}{\delta}\int_{\Omega_b} \boldsymbol{\psi}\cdot\mathbf{n}\,\mathrm{d}\Omega \to \int_{\Gamma_s} \boldsymbol{\psi}\cdot\mathbf{n}\,\mathrm{d}\Gamma \quad \text{when } \delta \to 0 \tag{13}$$

It now follows, by construction, that the strain rate $\dot{\boldsymbol{\varepsilon}}(\mathbf{x})$ can be expressed in its regularized form as

$$\dot{\boldsymbol{\varepsilon}} = \frac{1}{2}(\nabla\dot{\mathbf{u}} + \dot{\mathbf{u}}\nabla) = \begin{cases} \dot{\boldsymbol{\varepsilon}}_b \equiv \dot{\boldsymbol{\varepsilon}}_c + \dfrac{1}{2\delta}(\mathbf{n}[\dot{\mathbf{u}}] + [\dot{\mathbf{u}}]\mathbf{n}) & \text{if } \mathbf{x} \in \Omega_b \\ \dot{\boldsymbol{\varepsilon}}_c & \text{if } \mathbf{x} \in \Omega_c \end{cases} \tag{14}$$

where the continuous part $\dot{\boldsymbol{\varepsilon}}_c$ is given by

$$\dot{\boldsymbol{\varepsilon}}_c = \tfrac{1}{2}(\nabla\dot{\mathbf{u}}_c + \dot{\mathbf{u}}_c\nabla) \tag{15}$$

As to the stress rate $\dot{\boldsymbol{\sigma}}_b$ (corresponding to $\dot{\boldsymbol{\varepsilon}}_b$), we first note, because of continuity when δ is a 'small' measure, that

$$\dot{\boldsymbol{\varepsilon}}_c = \dot{\boldsymbol{\varepsilon}}_c(\mathbf{x}_-) = \dot{\boldsymbol{\varepsilon}}_c(\mathbf{x}_0) \tag{16}$$

where we have introduced the notation $\mathbf{x}_- \in \Omega_c$ such that

$$\mathbf{x}_- = \mathbf{x}_0 - \frac{\delta}{2}\mathbf{n}(\mathbf{x}_0), \quad \mathbf{x}_0 \in \Gamma_s \tag{17}$$

Upon introducing the expression for $\dot{\boldsymbol{\varepsilon}}_b$ into the constitutive equation (5), we now obtain with the continuity equation (16) that

$$\dot{\boldsymbol{\sigma}}_b(\mathbf{x}) = \dot{\boldsymbol{\sigma}}_c(\mathbf{x}_-) + [\dot{\boldsymbol{\sigma}}], \qquad [\dot{\boldsymbol{\sigma}}] = \dot{\boldsymbol{\sigma}}_b(\mathbf{x}_0) - \dot{\boldsymbol{\sigma}}_c(\mathbf{x}_-), \quad \forall \mathbf{x} \in \Omega_b \tag{18}$$

where

$$\boldsymbol{\sigma}_c = \mathbf{D}_c : \dot{\boldsymbol{\varepsilon}}_c, \qquad [\dot{\boldsymbol{\sigma}}] = [\mathbf{D}] : \dot{\boldsymbol{\varepsilon}}_c + \frac{1}{\delta}(\mathbf{D}_b\cdot\mathbf{n})\cdot[\dot{\mathbf{u}}] \tag{19}$$

In (19), we have also introduced the jump in stiffness $[\mathbf{D}] = \mathbf{D}_b - \mathbf{D}_c$, since $\mathbf{D}_b$ and $\mathbf{D}_c$ take, in general, different values ($\mathbf{D}^e$ or $\mathbf{D}^{ep}$) depending on the actual loading condition in Ω_b and Ω_c. These conditions are, pertinent to the strain driven format, given by

$$\mathbf{f} : \mathbf{D}^e : \dot{\boldsymbol{\varepsilon}}_c + \frac{1}{\delta}\mathbf{a}\cdot[\dot{\mathbf{u}}] > 0, \quad \mathbf{x} \in \Omega_b \tag{20a}$$

$$\mathbf{f} : \mathbf{D}^e : \dot{\boldsymbol{\varepsilon}}_c > 0, \quad \mathbf{x} \in \Omega_c \tag{20b}$$

We define the vectors $\mathbf{a}(\mathbf{n})$, as introduced in (20), and $\mathbf{b}(\mathbf{n})$ for later use, as follows:

$$\mathbf{a}(\mathbf{n}) = \mathbf{f}:\mathbf{D}^{e}\cdot\mathbf{n}, \qquad \mathbf{b}(\mathbf{n}) = \mathbf{n}\cdot\mathbf{D}^{e}:\mathbf{g} \tag{21}$$

Besides continuity of $\dot{\boldsymbol{\varepsilon}}_c$, as expressed in (16), it is necessary to require continuity of equilibrium, which implies continuity of the traction vector inside as compared to immediately outside Ω_b, i.e. $\mathbf{n}\cdot\dot{\boldsymbol{\sigma}}_c = \mathbf{n}\cdot\dot{\boldsymbol{\sigma}}_b$. In view of (18), this condition can be written as

$$\mathbf{n}\cdot[\dot{\boldsymbol{\sigma}}] = \mathbf{0} \tag{22}$$

We are now in a position to establish the localization condition from the equilibrium equation (22). We then combine (22) with (19b) to establish the localization condition

$$\frac{1}{\delta}\mathbf{Q}_b\cdot[\dot{\mathbf{u}}] = -\mathbf{n}\cdot[\mathbf{D}]:\dot{\boldsymbol{\varepsilon}}_c \tag{23}$$

In (23), $\mathbf{Q}_b$ is the acoustic tensor associated with $\mathbf{D}_b$, which will take the values $\mathbf{Q}^{ep}$ or $\mathbf{Q}^{e}$ (depending on whether plastic or elastic loading takes place), where

$$\mathbf{Q}^{e} = \mathbf{n}\cdot\mathbf{D}^{e}\cdot\mathbf{n}, \qquad \mathbf{Q}^{ep} = \mathbf{n}\cdot\mathbf{D}^{ep}\cdot\mathbf{n} = \mathbf{Q}^{e} - \frac{1}{h}\mathbf{b}\mathbf{a} \tag{24}$$

CONDITION FOR ONSET OF LOCALIZATION AT PLANE STRAIN

We consider the implications of the localization condition (23), due to the assumed loading conditions, (P) or (E), in Ω_b and Ω_c.

Case 1: Elastic unloading (E) is assumed in Ω_b as well as in Ω_c. With $\mathbf{D}_b = \mathbf{D}_c = \mathbf{D}^{e}$, we obtain

$$\frac{1}{\delta}\mathbf{Q}^{e}\cdot[\dot{\mathbf{u}}] = \mathbf{0} \tag{25}$$

Since $\mathbf{Q}^{e}$ is positive definite, we conclude that $[\dot{\mathbf{u}}] \equiv \mathbf{0}$ in (25). In fact, the solution $[\dot{\mathbf{u}}] \equiv \mathbf{0}$ is obtained whenever $\mathbf{Q}_b$ is positive definite.

Case 2: Plastic loading (P) is assumed in Ω_b as well as in Ω_c. With $\mathbf{D}_b = \mathbf{D}_c = \mathbf{D}^{ep}$, we obtain

$$\frac{1}{\delta}\mathbf{Q}^{ep}\cdot[\dot{\mathbf{u}}] = \mathbf{0} \tag{26}$$

It appears that the necessary condition for a non-trivial solution $[\dot{\mathbf{u}}] \neq \mathbf{0}$ is that $\mathbf{Q}^{ep}$ is singular, and this result is independent of the width δ of the regularized zone. This conclusion can be arrived at without involving a 'band' of finite width into the analysis; in fact, it is sufficient to consider a 'boundary', that is the singular surface, as described in the classical works of e.g. Thomas[8] and Rice.[7] However, since the width of the localization zone may be arbitrarily thin, the case of strong discontinuity appears as the extreme case.

It is thus of considerable interest to characterize the condition when $\mathbf{Q}^{ep}$ becomes singular. To this end, let us restrict the analysis to the condition for *plane strain*, whereby the smallest eigenvalue of $\mathbf{Q}^{ep}$ is obtained from the 'in-plane' eigenvalue problem

$$Q^{ep}_{\alpha\beta}y^{(i)}_{\beta} = \mu^{(i)}Q^{e}_{\alpha\beta}y^{(i)}_{\beta} \tag{27}$$

where Greek indices, which range from 1 to 2, define the in-plane components.

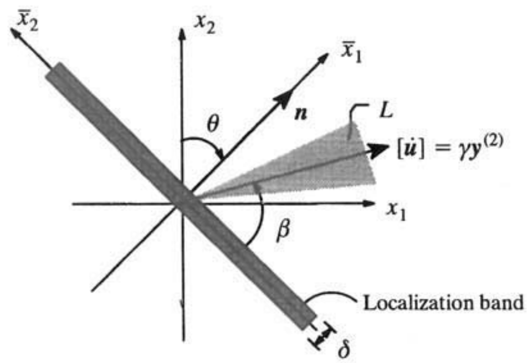

Figure 2. Band co-ordinates ($\bar{x}_a$) and principal stress co-ordinates (x_a).

It may be shown that the largest eigenvalue (corresponding to neutral plastic loading) is $\mu^{(1)} = 1$, whereas the smallest eigen value $\mu^{(2)}$ and the corresponding eigenvector $\mathbf{y}^{(2)}$ are given[9] as

$$\mu^{(2)} = 1 - \frac{1}{h}\mathbf{a} \cdot \mathbf{P}^{e} \cdot \mathbf{b}, \qquad \mathbf{y}^{(2)} = \gamma \mathbf{P}^{e} \cdot \mathbf{b} \tag{28}$$

Because of the plane strain condition, the band orientation is given by the in-plane unit vector $\mathbf{n} = n_a$. In particular, this direction is represented by the angle θ measured from the minor in-plane principal stress axis to the normal of the localization band, as shown in Figure 2.

For a given state (as expressed in terms of the internal state variables H, $\boldsymbol{\sigma}$, $\dot{\boldsymbol{\varepsilon}}$), it is possible to consider $\mu^{(2)}$ as a function of only the band orientation θ, i.e.

$$\mu^{(2)}_{\min} = \min \mu^{(2)}(\theta) \tag{29}$$

where $\mu^{(2)}_{\min}$ is the minimal value of $\mu^{(2)}(\theta)$ corresponding to the critical orientation θ^{cr}. We thus have

$$\theta^{cr} = \arg[\min \mu^{(2)}(\theta)] \tag{30}$$

As to the condition for the onset of localization, we argue as follows: In view of the minimization problem (29), the earliest possibility for singularity of the acoustic tensor occurs when $\mu^{(2)}_{\min} = 0$ has been attained for the first time. Hence, the condition that $\mu^{(2)}_{\min} = 0$ is considered as the proper condition for diagnosing the onset of localization. Moreover, the localization mode at incipient localization is determined by the eigenvector $\mathbf{y}^{(2)}(\theta^{cr})$, i.e. we have $[\dot{\mathbf{u}}] = \gamma \mathbf{y}^{(2)}(\theta^{cr})$, where γ is an arbitrary scalar. As an alternative, the localized mode is expressed in terms of the dilation angle β, as defined in Figure 2. From geometric considerations in this figure, we find that

$$\tan\beta = \tan\theta \, \frac{1 + (y^{(2)}_2/y^{(2)}_1)\cot\theta}{1 - (y^{(2)}_2/y^{(2)}_1)\tan\theta} \tag{31}$$

CONDITION FOR FURTHER LOCALIZATION AND ARREST

In this subsection, we discuss progressive development of localized deformation, as defined by the case when plastic loading (P) occurs in Ω_b, whereas elastic unloading (E) takes place in Ω_c. The

behaviour is thus considered beyond onset of localization for a band with fixed orientation (as defined by $\theta = \theta^{cr}$ relative to the current principal axes at the onset of localization).

The condition of plastic loading in Ω_b and elastic loading in Ω_c leads to

$$\frac{1}{\delta}\mathbf{Q}^{ep}\cdot[\dot{\mathbf{u}}] = \lambda_c\mathbf{b}, \qquad \lambda_c = \frac{1}{h}\mathbf{f}:\mathbf{D}^e:\dot{\boldsymbol{\varepsilon}}_c \tag{32}$$

where it is emphasized that all quantities in this expression, except $\dot{\boldsymbol{\varepsilon}}_c$, relates to the behavior of the band. As to the sign of λ_c, we note that λ_c may be negative or positive, since continuity of the stress state, inside as compared to outside the band, cannot be assured in the post-localized range, as discussed subsequently.

Nevertheless, since $(\mathbf{Q}^{ep})^{-1}$ can be found explicitly as

$$(\mathbf{Q}^{ep})^{-1} = \mathbf{P}^e + \frac{1}{h\mu^{(2)}}\mathbf{P}^e\cdot\mathbf{ba}\cdot\mathbf{P}^e \quad \text{with} \quad \mathbf{P}^e = (\mathbf{Q}^e)^{-1} \tag{33}$$

we may solve for $[\dot{\mathbf{u}}]$ from (32) to obtain

$$[\dot{\mathbf{u}}] = \gamma\mathbf{P}^e\cdot\mathbf{b}, \qquad \gamma = \frac{\delta}{\mu^{(2)}}\lambda_c \tag{34}$$

It is noted that this solution is represented by the eigenmode $\mathbf{y}^{(2)}$ that corresponds to the smallest eigenvalue $\mu^{(2)}$, as defined in (28). It may also be of some interest to consider the discontinuity of the stress rate, which is given by the constitutive relation (19b). Upon introducing the current loading condition, i.e. $\mathbf{D}_b = \mathbf{D}^{ep}$ and $\mathbf{D}_c = \mathbf{D}^e$, and invoking (34) into (19b), we obtain

$$[\dot{\boldsymbol{\sigma}}] = -\frac{\lambda_c}{\mu^{(2)}}(\mathbf{D}^e:\mathbf{g} - (\mathbf{D}^e\cdot\mathbf{n})\cdot(\mathbf{P}^e\cdot\mathbf{b})) \tag{35}$$

where it may be noted that $[\dot{\boldsymbol{\sigma}}] \neq \mathbf{0}$ in general, whereas traction continuity is always ensured. That is, in view of the definition of the quantities $\mathbf{b}$ and $\mathbf{P}^e$, we always obtain the desired condition that

$$\mathbf{n}\cdot[\dot{\boldsymbol{\sigma}}] \equiv \mathbf{0}.$$

Let us next consider the validity of the solution (34) on the basis of the condition for plastic loading in Ω_b, which, in view of (20) can be rephrased as

$$\lambda_c + \frac{1}{\delta h}\mathbf{a}\cdot[\dot{\mathbf{u}}] \equiv \frac{\lambda_c}{\mu^{(2)}} > 0 \quad \text{in } \Omega_b \tag{36}$$

where the solution in (34) was invoked to give the last inequality in (36).

Assuming that $\lambda_c < 0$, which means 'elastic unloading' in Ω_c relative to the plastic properties in Ω_c, we obtain from (36) that $\mu^{(2)} < 0$. On the other hand, if $\lambda_c > 0$, which means 'elastic loading' in Ω_c relative to the plastic properties in Ω_c, then plastic loading in Ω_b can only be obtained provided at $\mu^{(2)} > 0$. The latter situation corresponds to 'stress increase', inside as well as outside the bond, and the considered loading assumption be accomplished as long as $F(\boldsymbol{\sigma}_c) < 0$.

Remark: The assumption of plastic loading in Ω_b and elastic unloading in Ω_c also pertains to the onset of localization. In that situation the stress state is continuous throughout the solid, whereby it must be required that $\lambda_c < 0$ (corresponding to elastic unloading in Ω_c) in order to satisfy the loading assumption. This gives the condition that $\mu^{(2)} < 0$ from (36).

Evidently, the loading condition that (P) occurs in Ω_b and (E) occurs in Ω_c can only be maintained without any restrictions as long as $\mu^{(2)} < 0$. As opposed to the situation at the onset of localization, we are now concerned with a band of fixed orientation, which means that the value of $\mu^{(2)}$ is considered with respect to changes of the state variables. In particular, we consider the effect on $\mu^{(2)}$ that is caused by changing the stress state due to continued loading beyond incipient localization, cf. Figure 2. To be more explicit, we may consider the situation that the principal axes can rotate for a given ratio of (in plane) principal stress values. It appears that this is the reverse problem as compared to the problem of finding the critical and orientation at the onset of localization, as discussed in Reference 10.

Hence, we conclude that the localization band may develop further only if $\mu^{(2)}(\theta) < 0$. This situation is, in particular, obtained for the deformation mode $[\dot{\mathbf{u}}] = \gamma \mathbf{P}^e \cdot \mathbf{b}$. If, on the other hand, $\mu^{(2)}(\theta) \geqslant 0$, then the deformation process is 'hardening' (for all possible modes $[\dot{\mathbf{u}}]$), and further localization can take place only if $F(\boldsymbol{\sigma}_c) < 0$. Whenever $F(\boldsymbol{\sigma}_c) > 0$ is predicted, plastic loading is obtained outside the band and the only solution is that $[\dot{\mathbf{u}}] \equiv \mathbf{0}$, which is a situation of arrested localization.

It appears that continuing localization, i.e. $\mu^{(2)}(\theta) < 0$, is possible within a 'fan' L of principal stress directions, as depicted in Figure 2. It should be noted that L is generally non-empty when $\mu^{(2)}_{\min} < 0$.

MODEL BEHAVIOR BASED ON THE MOHR–COULOMB CRITERION

A simple, but yet still representative, Mohr–Coulomb model will be considered within the frame-work outlined above. The main purpose of the model is to describe the non-linear stress–strain relationship towards shear failure. Similar models have previously been analyzed, for example, by de Borst[11] and Leroy and Ortiz.[12] We thus define the yield criterion F and the flow potential G as

$$F(p, q, \varkappa) = q - \eta(\varkappa)(p - p_c) \tag{37a}$$

$$G(p, q) = q - \mu(p - p_c) \tag{37b}$$

where p_c is the cohesion; $\eta = \sin\phi$ (ϕ = friction angle) and $\mu = \sin\phi$ (ψ = dilation angle) are friction and (plastic) dilation parameters, respectively. Moreover, the invariants p and q in (37) are the (in-plane) effective stress deviator q and the (in-plane) pressure p. These are related to the 'current' principal values of the (total) stress tensor $\boldsymbol{\sigma}$ as

$$q = \tfrac{1}{2}(\sigma_1 - \sigma_3), \qquad \sigma_1 \geqslant \sigma_2 \geqslant \sigma_3 \tag{38a}$$

$$p = -\tfrac{1}{2}(\sigma_1 + \sigma_3) \tag{38b}$$

The function $\eta(\varkappa)$ in (37) is described by

$$\eta(\varkappa) = \begin{cases} \eta(0) + 2\dfrac{\sqrt{\varkappa/\varkappa_f}}{1 + \varkappa/\varkappa_f}(\bar{\eta} - \eta(0) & \text{if } \varkappa < \varkappa_f \\ \eta(\varkappa_f) = \bar{\eta} & \text{if } \varkappa \geqslant \varkappa_f \end{cases} \tag{39}$$

where $\bar{\eta} = \eta(\varkappa_f)$ is the peak value. We also define the evolution rule for the hardening variable $\varkappa$ as

$$\dot{\varkappa} = \dot{\lambda}\frac{\partial G(p, q)}{\partial q} = \dot{\lambda} \tag{40}$$

which means that $\varkappa$ is associated with the plastic portion of the effective (in-plane) shear strain. In accordance with the notation introduced above, we now obtain hardening or perfectly plastic behavior from the relationship

$$H = \begin{cases} \dfrac{\bar{\eta} - \eta_i}{\varkappa_f} \sqrt{\dfrac{\varkappa^f}{\varkappa}} \dfrac{1 - \varkappa/\varkappa_f}{(1 + \varkappa/\varkappa_f)^2} (p - p_c) > 0 & \text{if } \varkappa < \varkappa_f \\ 0 & \text{if } \varkappa \geqslant \varkappa_f. \end{cases} \tag{41}$$

Localization properties based on isotropic elasticity and plane strain

We shall evaluate the behaviour of the localized zone due the present Mohr–Coulomb model. The analysis is then restricted to isotropic elasticity, as defined by the shear modulus G and Poisson's ratio v. Moreover, the plane strain condition is considered (as indicated previously). With these assumptions, we find that $\mathbf{Q}^e$ and its inverse $\mathbf{P}^e$ can be written

$$\mathbf{Q}^e_{\alpha\beta} = G\left(\delta_{\alpha\beta} + \frac{1}{v^*} n_\alpha n_\beta\right), \qquad \mathbf{P}^e_{\alpha\beta} = \frac{1}{G}\left(\delta_{\alpha\beta} - \frac{1}{1 + v^*} n_\alpha n_\beta\right) \tag{42}$$

where $v^* = 1 - 2v$.

In order to obtain a convenient structure of the equations, we shall henceforth consider the in-plane principal stress axes x_1 and x_2, which are labeled such that the corresponding in-plane principal stress $\sigma_1 > \sigma_2$. In this coordinate system, the vectors $\mathbf{a}$ and $\mathbf{b}$ are obtained as

$$\mathbf{a}_\gamma = \frac{2G}{v^*} a_\gamma n_\gamma, \qquad b_\gamma = \frac{2G}{v^*} \beta_\gamma n_\gamma \quad \text{(no sum)} \tag{43}$$

where it was tacitly assumed that the intermediate principal stress σ_2, as defined in (38a), is associated with the out-of-plane direction. Moreover, in (43), we introduced the coefficients a_γ and β_γ, which are defined as

$$a_1 = v^* + \eta, \qquad a_2 = -(v^* - \eta)$$

$$\beta_1 = v^* + \mu, \qquad \beta_2 = -(v^* - \mu)$$

We also obtain the plastic modulus as

$$h = G\left(1 + \frac{\eta\mu}{v^*}\right)\frac{1}{1 - (G_T/G)} \tag{44}$$

where the ratio G_T/G, which is introduced for the sake of convenience, is defined via

$$\frac{G_T}{G} = \frac{H/G}{1 + (\eta\mu/v^*) + (H/G)} \tag{45}$$

Upon introducing (42b), (43) and (44) into (28), the eigenvalue $\mu^{(2)}(\theta)$ can now be written as

$$\mu^{(2)}(\theta) = 1 - \left(1 - \frac{G_T}{G}\right)\frac{1}{v^*(1 + v^*)(v^* + \eta\mu)}\Big[\sin^2\theta((1 + v^*)\alpha_1\beta_1 - \beta_1 r(\theta))$$

$$+ \cos^2\theta((1 + v^*)a_2\beta_2 - \beta_2 r(\theta))\Big] \tag{46}$$

where $r(\theta) = a_1 \sin^2\theta^{cr} + a_2\cos^2\theta^{cr}$.

Condition for the onset of localization

The most critical situation, for which $\mu^{(2)}(\theta)$ is minimum with respect to variations of the orientation θ, is obtained from the minimization problem (29). By application of (46) to (29), we obtain the critical orientation

$$\tan^2 \theta^{cr} = \frac{2 + \mu + \eta}{2 - \mu - \eta} \tag{47}$$

corresponding to the minimal eigenvalue

$$\mu^{(2)}_{\min} = \left(\frac{G_T}{G} - \frac{1}{1 + (\eta\mu/v^*)}\left(1 - \frac{G_T}{G}\right)\frac{(\eta - \mu)^2}{4(1 + v^*)}\right) \tag{48}$$

(a)

(b)

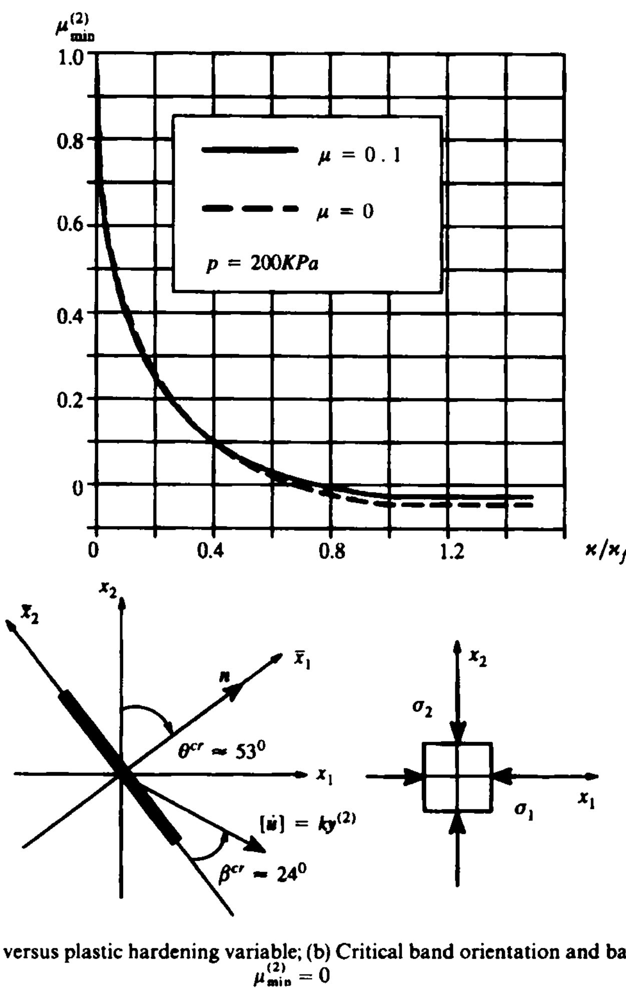

Figure 3. (a) Decay of $\mu^{(2)}_{\min}$ versus plastic hardening variable; (b) Critical band orientation and band dilation angle when $\mu^{(2)}_{\min} = 0$

In order to assess the dilation inside the band at the onset of localization, we obtain, from the eigenvector in (28b) and from (31), the dilation at onset of localization

$$\tan\beta^{cr} = \frac{1}{\alpha_1 - \alpha_2}\left(\left(\alpha_1 - \frac{r(\theta^{cr})}{1+v^*}\right)\tan\theta^{cr} + \left(\alpha_2 - \frac{r(\theta^{cr})}{1+v^*}\right)\cot\theta^{cr}\right) \tag{49}$$

where it is noted that the localized mode at the onset of localization, generally, will consist of both shear and normal components relative to the critical band orientation, i.e. $\beta^{cr} \neq 90°$.

As an example, we shall consider the value of $\mu^{(2)}_{min}$ as a function of the amount of plastic deformation (represented by the hardening variable $\varkappa$). To this end, the confining pressure is taken as $p = 200$ KPa, whereas the material properties are: $E = 66\,000$ KPa, $v = 0{\cdot}3$, $\eta(0) = 0{\cdot}1$, $\bar{\eta} = 0{\cdot}5$, $\varkappa_f = 0{\cdot}01$, and $p_c = -1$ KPa.

The results are shown in Figure 3(a) when the dilation angle is chosen as $\mu = 0$ and $\mu = 0{\cdot}1$. As expected, due the non-associativity, singularity of the acoustic tensor is obtained in the hardening

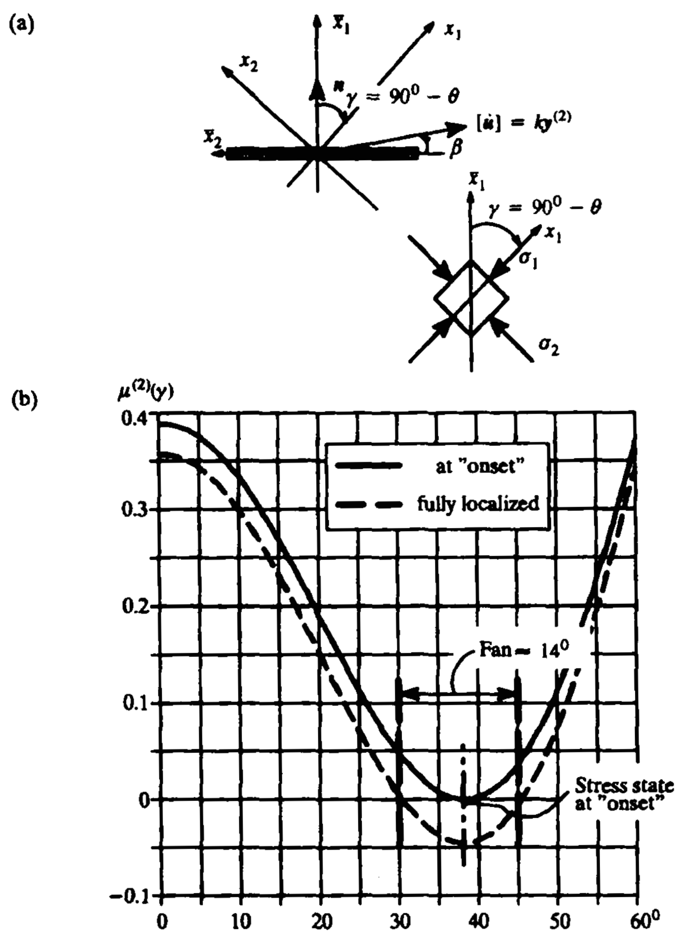

Figure 4. (a) Variation of state inside localization band by rotation in-plane stress axes; (b) Response in localization band in post-localized range when $\mu = 0$

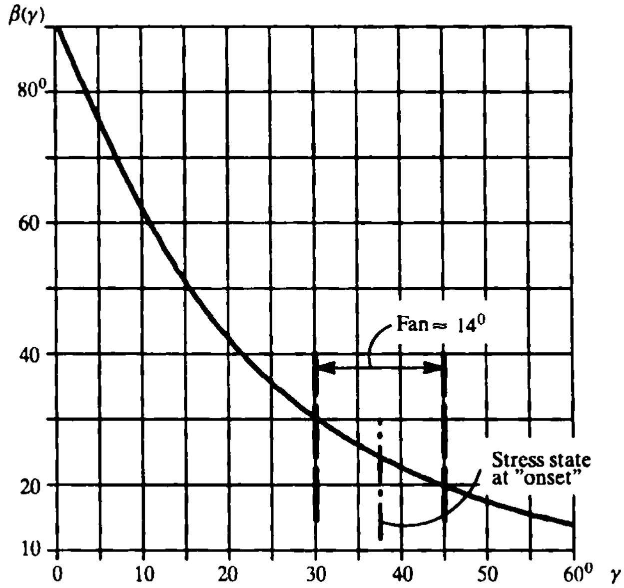

Figure 5. Dilation angle in localization band at fully localized stage range when $\mu = 0$

regime. Moreover, it appears that in-plane plastic incompressibility is most sensitive to loss of singularity and development of localized deformation, i.e. singularity is obtained for $\varkappa/\varkappa_f = 0{\cdot}68$ when $\mu = 0$ and $\varkappa/\varkappa_f = 0{\cdot}76$ for $\mu = 0{\cdot}1$. As to the critical angles of band orientation and plastic dilation, we obtain at the point where loss of singularity occurs that $\theta^{cr} = 53^\circ$ and the band dilation angle is $\beta^{cr} \approx 24^\circ$ for both choices of μ, as shown in Figure 3(b).

Remark: The present represents a case of 'ductile localization' in the sense that the state corresponding to $\mu^{(2)}_{min} = 0$ is actually attained at continuous loading along a smooth portion of the $\mu^{(2)}_{min}$ versus $\varkappa$ curve, as shown in Figure 3(a). Localization is thus preceded by considerable plastic deformation, which is the classical situation at the development of a shear band (governed by the von Mises or Tresca yield criterion).

Behavior in post-localized range

Next, the fan L is investigated in the post-localized range for a band of fixed orientation. In particular, we are concerned with the range of possible rotations $\gamma = 90^\circ - \theta$ (Figure 4(a)) of the principal stress axes relative to the band with respect to the condition that $\mu^{(2)}(\theta) < 0$. We use the same material properties as in the evaluation of the onset of localization.

The result shown in Figure 4(b) considers two different stages during the localization process. At the first stage, the model behaviour is evaluated right at the very onset of localization. It appears that the fan L is empty at this stage and the only feasible localization mode is $\beta^{cr} \approx 24^\circ$. At the second stage, the band has been fully localized (defined by $\varkappa > \varkappa_f$). The resulting fan is now $L \approx 14^\circ$ centered on the stress state at the onset of localization. Within this fan, the localization mode may vary between $\beta \approx 20^\circ$ and 30°, as shown in Figure 5.

CONCLUDING REMARKS

The localization properties pertinent to the present Mohr–Coulomb model were evaluated with respect to the existence and development of a regularized strong discontinuity. The constitutive behaviour of the arising localization band was thereby investigated with respect to onset of localization, band orientation as well as behavior in the post-localized range.

It was shown that the condition of the existence of such a discontinuity (= onset of localization) is obtained when the smallest eigenvalue of the acoustic tensor is zero, and this result is independent of the choice of regularization width δ. For the present model, the localization trigger is the degree of non-associativity, as determined y the dilatancy parameter. The earliest possibility for localization was obtained for in-plane plastic incompressibility ($\mu = 0$).

In the post localized range, with the band orientation being fixed at the onset of localization, the localization band continues to develop in a softening fashion, for continued loading, when the principal stress axes (relative to the band orientation) are within the fan L. This fan was shown to be empty at the onset of localization, since the condition that $\mu = 0$ is, infact, attained. This is a principal difference from the behavior of a brittle fracture process, as discussed by Larsson and Runesson.[10] If localization develops, the fan L increases and the response of the localization band becomes more flexible until the fully localized stage has been developed. The predicted amount of softening is also, partly, determined by δ, in the sense that the strength of the discontinuity depends on δ, and the band 'softening', which is induced by the degree of non-associativity. Therefore, since the dilatancy and frictional parameters are fixed, the only parameter to work with is δ at the clibration of the post-localized response. Hence, the regularization width should be regarded as a 'material parameter' for the behavior in the post-localized response.

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